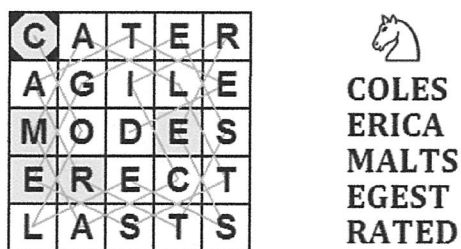


WORD SQUARES WITH WORDY KNIGHT'S TOURS

MIKE KEITH
New Jersey, USA

The word square displayed below has a remarkable property. On the surface it is an ordinary double word square, with a different word in each row and column. In addition, starting in the upper left corner and following the knight's tour indicated by the lines reveals another five 5-letter words (listed on the right, below the knight symbol). All fifteen words are in both *Webster's 3rd Unabridged* and the North American Scrabble *Tournament and Club Word List* (TWL06). The shaded squares highlight the first letter of each word on the knight's tour, while the octagon indicates the start of the tour.



How many other words squares like this are there? In what sizes do they come? Which ones have special properties, such as having many distinct letters? What about single word squares?

To answer these questions we start with some basics, beginning with the question of size. A knight's tour on an $n \times n$ board is only possible for $n \geq 5$, so squares of this kind that are 4×4 or smaller cannot exist. So let's consider the 5×5 case first. We can use a computer program to exhaustively find all 5×5 word squares that can be constructed using the 8938 five-letter words in the TWL06 word list, allowing both double squares (where the words in the rows and columns are different) and single squares (where the five words in the rows are replicated in the columns). We find that there are 39,629,727 different double squares and 3,558,563 single squares, or 43,188,290 total squares. Note that for single (resp., double) squares, all five (resp., ten) words in the square are required to be different. Also, any double square that is the transpose of another solution is not counted.

The number of distinct directed open knight's tours on an $n \times n$ board for $n \geq 5$ [1] is

1728, 6637920, 16557521830, 19591828170979904, ...

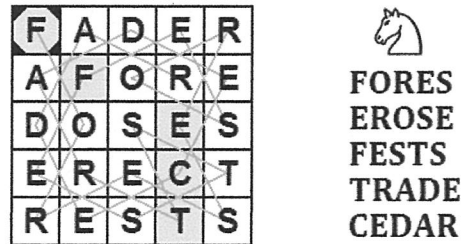
To find all possible 5×5 squares of this kind we "simply" have to try each of the 1728 tours on each of the 43,188,290 squares and find which combinations give a tour spelling out five 5-letter words like the square shown above. A quick estimate suggests that exhaustively testing all 74,629,365,120 cases using an efficient computer program can be done in a reasonable amount of execution time, so we wrote a computer program to do just that. After four hours of runtime on an ordinary desktop PC we find that:

There are 484 different double word squares having a wordy knight's tour. A few of these squares allow more than one wordy tour, so these 484 squares actually generate a total of 510 solutions (combination of letter grid and knight's tour).

There are 25 different single word squares of this type. Multiple tours are more common for single squares; if these are counted then there are a total of 80 solutions.

The square shown at the beginning of this article is one of our favorite double-square solutions, as only a few of its 15 words are uncommon (COLES are cabbages, ERICA is a type of heather, EGEST is the opposite of ingest, and AGORA is, well, the thing that scares agoraphobes).

An example of a single word square of this type is shown below.



From here on we consider double squares only. Let L be the number of distinct letters of the alphabet in the grid. All values of L from 9 to 16 appear in the set of solutions. A square with 9 different letters (AEILNRSVX) and the unique square with 16 different letters (ABCDEIKLMNORSTVW) are shown below.



9 different letters



16 different letters



Oddly enough, two of the nine letters in the $L=9$ square are V and X. What about rare letters in general? We find that there are 5 solutions with a J, none with Q, 27 with an X, and 14 with Z. There are even two solutions that contain both X and Z:



The $L=16$ square shown above has four of the shaded squares in a line. It is possible for all five to be in a line? The answer is yes; here are the only two squares with this property:

A	P	A	R	T
M	A	N	E	S
A	L	O	H	A
S	E	D	E	R
S	T	E	M	S


SHAME
MARAS
EROSE
TAPED
SLANT

B	A	R	E	R
A	N	E	L	E
L	I	P	I	D
E	M	O	T	E
R	A	S	E	S


RISEN
EDEMA
REPEL
ALATE
BIROS

If the 5x5 board is colored in the standard chessboard fashion, with the corner squares black, then there are 13 black squares and 12 white squares. Since a knight's tour alternates colors on each move it is easy to see that any tour must start and end on a black square. Does a solution exist for each possible starting square? Yes, and the diagram below shows the number of solutions starting in each black square. The sum of these numbers is 510, the total number of solutions.

184		30		83
	14		7	
58		13		15
	9		2	
64		5		26

One key characteristic of squares of this kind is the arrangement of consonants and vowels. If, for example, every black square on the chessboard contains a consonant and every white square a vowel then it will tend to be amenable to making words in the rows, columns *and* on the knight's tour, since the knight's tour alternates colors (as do the rows and columns). In fact, in the first version of our search program we enforced this restriction in order to make the program run faster, on the theory that if there were any solutions to this puzzle then surely *some* of them would have strictly alternating consonants and vowels. Indeed, it turns out that 267 of the 484 grids (about 55%) have this strictly-alternating pattern of consonants and vowels:

C	V	C	V	C
V	C	V	C	V
C	V	C	V	C
V	C	V	C	V
C	V	C	V	C

Figure 1. The most common consonant/vowel distribution

Let D be the number of squares in a grid that *don't* match this pattern. Table 1 shows the number of solution grids for all possible values of D, which range from D=0 (for the 267 perfectly-alternating grids) to D=10. Note that for simplicity we always classify Y as a vowel.

Value of D	0	1	2	3	4	5	6	7	8	9	10
No. of grids	267	122	100	9	3	-	-	-	2	-	1

The most unusual consonant/vowel distribution is shown below, in the unique solution with D = 10:

S	M	A	S	H
L	A	N	C	E
A	C	T	O	R
T	H	E	R	E
Y	E	S	E	S



HOSEY
CREAM
CESTA
STEAL
HERNS

The 10 letters that don't match the pattern of Figure 1 are the MAS in SMASH, the LAN in LANCE, the AC in ACTOR, and the first letter of THERE and YESES.

We mentioned earlier that a few double-square solutions contain multiple wordy tours. In fact, there is one square that allows *three* different tours:

T	I	M	E	R
A	R	E	T	E
L	O	T	U	S
A	N	O	D	E
R	E	L	E	T



MATED
ELITE
LARES
ENURE
TOROT

T	I	M	E	R
A	R	E	T	E
L	O	T	U	S
A	N	O	D	E
R	E	L	E	T



TAMED
ELITE
LARES
ENURE
TOROT

T	I	M	E	R
A	R	E	T	E
L	O	T	U	S
A	N	O	D	E
R	E	L	E	T



TOROT
ELITE
LARES
ENATE
DEMUR

Admittedly, the second tour is a minor variation of the first (it reverses the order of traversal of the first three squares), and the third tour has three words in common with the other tours. Nevertheless, we think this square is quite remarkable.

Does a 6x6 square of this kind exist using TWL06 words? The number of 6x6 double word squares constructible with TWL06 is about 1/200th the number of 5x5 squares, and the odds of a tour accidentally spelling six 6-letter words is significantly smaller than the odds of spelling five 5-letter words. On the other hand, the number of 6x6 tours is about 4000 times larger than the number of 5x5s. We tentatively conjecture that there are no 6x6 solutions but this remains an open question. One tantalizing possibility is that there may be a 6x6 square having a knight's tour that spells nine 4-letter words (rather than six 6-letter words).

References

[1] Sequence A165134 in the *Online Encyclopedia of Integer Sequences* (oeis.org).